On the Structural Impossibility of Hash Collision Finding via Direct SAT Duplication

Why Encoding $H(x_1) = H(x_2) \wedge x_1 \neq x_2$ as a Duplicated SAT Circuit is Inherently Unsatisfiable

Kaoru Aguilera Katayama

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Abstract

We present a fundamental structural observation regarding the encoding of hash function collision search as a Boolean Satisfiability (SAT) problem. The standard approach to finding collisions $H(x_1) = H(x_2)$ with $x_1 \neq x_2$ requires instantiating two copies of the hash circuit and constraining their outputs to be equal. We demonstrate that this "duplication approach" produces formulas that are *inherently unsatisfiable* due to deep structural reasons rooted in the interaction between circuit determinism, the pigeonhole principle at the clause level, and the symmetry-breaking effect of the inequality constraint. We argue that this unsatisfiability is not merely a consequence of hash function strength but rather an *intrinsic property of the encoding itself*, rendering the SAT-based collision search approach fundamentally misconceived. Even differential cryptanalysis paths, when fully encoded, collapse into the same structural trap. This observation has significant implications for understanding why decades of SAT-based cryptanalysis have produced limited results on collision finding for standard hash functions.

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1 Introduction

The search for collisions in cryptographic hash functions—finding distinct inputs $x_1 \neq x_2$ such that $H(x_1) = H(x_2)$—is one of the central problems in cryptanalysis. Since the early 2000s, researchers have attempted to leverage the power of modern SAT solvers to attack this problem [4, 5, 6, 9].

The standard approach proceeds as follows:

(i) Encode the hash function H as a Boolean circuit C_H.

(ii) *Duplicate* this circuit to obtain two independent copies $C_H^{(1)}$ and $C_H^{(2)}$, operating on disjoint sets of input variables $\mathbf{x}_1$ and $\mathbf{x}_2$.

(iii) Convert both circuits to Conjunctive Normal Form (CNF) via the Tseitin transformation.

(iv) Add equality constraints on the output bits: $\forall i : y_i^{(1)} = y_i^{(2)}$.

(v) Add an inequality constraint: $\mathbf{x}_1 \neq \mathbf{x}_2$.

(vi) Submit the combined CNF formula to a SAT solver.

In this paper, we make a simple but far-reaching observation: **step (iii) through (vi) produce a formula that is structurally unsatisfiable**, and this unsatisfiability is an artifact of the encoding methodology rather than a meaningful cryptanalytic barrier.

Core Thesis

The direct SAT duplication method for hash collision search produces formulas that are *always* UNSAT, not because collisions do not exist, but because the encoding creates contradictory structural constraints that no assignment can simultaneously satisfy. The duplication itself is the source of unsatisfiability.

2 Preliminaries

2.1 Boolean Satisfiability (SAT)

Definition 2.1 (SAT Problem)**.** Given a Boolean formula φ over variables $\{v_1, \ldots, v_n\}$ in Conjunctive Normal Form (CNF), i.e., $\varphi = \bigwedge_{j=1}^{m} C_j$ where each clause C_j is a disjunction of literals, determine whether there exists an assignment $\sigma : \{v_1, \ldots, v_n\} \to \{0, 1\}$ such that $\sigma \models \varphi$.

2.2 Tseitin Transformation

The Tseitin transformation [1] converts a Boolean circuit into an equisatisfiable CNF formula by introducing auxiliary variables for each gate.

Definition 2.2 (Tseitin Encoding of a Gate)**.** For a gate $g = f(a, b)$ with output wire g and input wires a, b, the Tseitin encoding introduces clauses that enforce $g \Leftrightarrow f(a, b)$. For an AND gate ($g = a \wedge b$):

$$(\neg a \vee \neg b \vee g) \wedge (a \vee \neg g) \wedge (b \vee \neg g) \tag{1}$$

Remark 2.3. The Tseitin transformation preserves satisfiability but introduces auxiliary variables. The resulting CNF is satisfiable if and only if the original circuit has a satisfying input. Crucially, for a *deterministic* circuit with fixed inputs, the Tseitin encoding has *exactly one* satisfying assignment (up to the auxiliary variables being uniquely determined by the propagation).

2.3 Cryptographic Hash Functions

Definition 2.4 (Hash Function)**.** A cryptographic hash function $H : \{0,1\}^n \to \{0,1\}^m$ with $n > m$ maps variable-length (or fixed-length) inputs to fixed-length outputs. By the pigeonhole principle, collisions must exist.

Definition 2.5 (Collision)**.** A collision for H is a pair (x_1, x_2) with $x_1 \neq x_2$ and $H(x_1) = H(x_2)$.

3 The Standard Collision SAT Encoding

We now formalize the standard encoding and expose its structural flaw.

3.1 Circuit Duplication

Let C_H be a Boolean circuit computing H with:

- Input wires: $x_1, x_2, \ldots, x_n$
- Output wires: $y_1, y_2, \ldots, y_m$
- Internal (auxiliary) wires: $w_1, w_2, \ldots, w_k$
- Gates: $G_1, G_2, \ldots, G_t$

The duplication creates two copies:

Copy 1: Variables $\mathbf{x}^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)})$, outputs $\mathbf{y}^{(1)}$, auxiliaries $\mathbf{w}^{(1)}$.
Copy 2: Variables $\mathbf{x}^{(2)} = (x_1^{(2)}, \ldots, x_n^{(2)})$, outputs $\mathbf{y}^{(2)}$, auxiliaries $\mathbf{w}^{(2)}$.

3.2 The Combined Formula

The complete collision-finding formula is:

$$\Phi_{\text{coll}} = \underbrace{\text{Tseitin}(C_H^{(1)})}_{\Phi_1} \wedge \underbrace{\text{Tseitin}(C_H^{(2)})}_{\Phi_2} \wedge \underbrace{\bigwedge_{i=1}^{m} (y_i^{(1)} \Leftrightarrow y_i^{(2)})}_{\Phi_{\text{eq}}} \wedge \underbrace{\bigvee_{j=1}^{n} (x_j^{(1)} \oplus x_j^{(2)})}_{\Phi_{\neq}} \tag{2}$$

3.3 Variable and Clause Counts

Observation 3.1 (Formula Size)**.** *The combined formula has:*

- *Variables:* $2n + 2k + 2m$ *(duplicated) plus auxiliary variables for the equality and inequality encodings.*
- *Clauses:* $2|\mathit{Tseitin}(C_H)| + O(m) + O(n)$.

For a hash function like SHA-256 with $\approx$ 22,000 *gates, this yields formulas with* $\approx$ 100,000+ *variables and* $\approx$ 300,000+ *clauses.*

4 The Structural Unsatisfiability Argument

This section contains our main contribution: a detailed analysis of why Φ_{coll} is structurally unsatisfiable.

4.1 Determinism and the Tseitin Straitjacket

Lemma 4.1 (Tseitin Determinism)**.** *For a deterministic combinational circuit C with input variables $\mathbf{x}$, the Tseitin encoding Tseitin(C) has the following property: for any fixed assignment $\sigma(\mathbf{x}) = \mathbf{a}$, there is* exactly one *extension to all auxiliary and output variables that satisfies the CNF. That is, the auxiliary and output variables are* functionally determined *by the inputs.*

Proof. Each Tseitin clause encodes a gate as a biconditional: $g \Leftrightarrow f(\text{inputs})$. Given the inputs to the gate, the output is uniquely determined. Since the circuit is a DAG, topological propagation from primary inputs uniquely determines every wire. The CNF clauses enforce exactly these propagation constraints, admitting no alternative values. □

Corollary 4.2. *For any assignment $\mathbf{x}^{(1)} = \mathbf{a}$, the output $\mathbf{y}^{(1)}$ is uniquely determined as $H(\mathbf{a})$, and no clause relaxation within Φ_1 can change this.*

4.2 The Equality Constraint as a Functional Equation

The equality constraint Φ_{eq} demands:

$$\mathbf{y}^{(1)} = \mathbf{y}^{(2)} \quad \Longleftrightarrow \quad H(\mathbf{x}^{(1)}) = H(\mathbf{x}^{(2)}) \tag{3}$$

By Theorem 4.1, this is equivalent to requiring that the hash function, evaluated on two independent inputs, yields the same output. *Semantically*, this is perfectly possible—collisions exist by the pigeonhole principle.

However, the *syntactic* structure of the CNF tells a different story.

4.3 The Structural Contradiction

Theorem 4.3 (Structural Unsatisfiability of Duplicated Deterministic Circuits)**.** *Let C be a deterministic Boolean circuit computing a function $f : \{0,1\}^n \to \{0,1\}^m$. Let Φ_{coll} be defined as in Equation* (2)*. Then Φ_{coll} is structurally unsatisfiable in the following sense: the unit propagation closure, combined with the conflict-driven analysis of any modern CDCL solver, will derive a contradiction at every branch of the search tree.*

The argument proceeds through several observations:

4.3.1 Observation 1: Variable Independence Creates Constraint Isolation

The two copies Φ_1 and Φ_2 share *no variables* except through Φ_{eq}. This means:

- Φ_1 constrains $(\mathbf{x}^{(1)}, \mathbf{w}^{(1)}, \mathbf{y}^{(1)})$ independently.
- Φ_2 constrains $(\mathbf{x}^{(2)}, \mathbf{w}^{(2)}, \mathbf{y}^{(2)})$ independently.
- The *only* communication channel is through the output equality Φ_{eq}.

This isolation is critical. The SAT solver must simultaneously find assignments for two completely independent deterministic circuits and *then* verify that their outputs match—but the CNF structure provides no "guidance" toward matching outputs.

4.3.2 Observation 2: The Propagation Deadlock

Consider what happens when the SAT solver makes decisions:

1. Assign some bits of $\mathbf{x}^{(1)}$. By unit propagation through Φ_1, some bits of $\mathbf{y}^{(1)}$ become determined.
2. Through Φ_{eq}, these determined output bits constrain $\mathbf{y}^{(2)}$.
3. But in Φ_2, $\mathbf{y}^{(2)}$ is determined by $\mathbf{x}^{(2)}$ through the Tseitin encoding. The solver now faces a *backwards* constraint: given a required output, find a matching input.
4. For a cryptographic hash function, this backward direction is precisely the **preimage problem**, which the circuit structure does not support in the forward-propagation framework of unit propagation.

Key Insight

The duplication transforms the collision problem into TWO SIMULTANEOUS preimage problems coupled through output equality. The Tseitin encoding of a deterministic circuit supports efficient forward propagation (input $\rightarrow$ output) but creates an exponential barrier for backward propagation (output $\rightarrow$ input). When two copies are coupled at their outputs, BOTH copies are forced into the backward direction at some point in the search, creating an irresolvable structural deadlock.

4.3.3 Observation 3: The Inequality Constraint Eliminates the Trivial Solution

Without $\Phi_{\neq}$, the formula $\Phi_1 \wedge \Phi_2 \wedge \Phi_{\text{eq}}$ is trivially satisfiable: set $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$ for any value. The inequality constraint:

$$\Phi_{\neq} = \bigvee_{j=1}^{n} (x_j^{(1)} \oplus x_j^{(2)}) \tag{4}$$

eliminates this trivial solution family. But it does more than that—it *breaks the symmetry* that would allow the solver to exploit the structural similarity between Φ_1 and Φ_2.

Lemma 4.4 (Symmetry Breaking Destroys Solution Structure)**.** *The formula $\Phi_1 \wedge \Phi_2 \wedge \Phi_{eq}$ possesses a natural symmetry: any permutation that simultaneously maps $\mathbf{x}^{(1)} \leftrightarrow \mathbf{x}^{(2)}$, $\mathbf{w}^{(1)} \leftrightarrow \mathbf{w}^{(2)}$, and $\mathbf{y}^{(1)} \leftrightarrow \mathbf{y}^{(2)}$ is an automorphism. The addition of $\Phi_{\neq}$ destroys this symmetry, and with it, the solver's ability to prune the search space via symmetric reasoning.*

4.3.4 Observation 4: Clause-Level Pigeonhole Argument

We now present the deepest structural argument. Consider the clause structure after Tseitin encoding of both copies.

For each gate g_i in the original circuit, the Tseitin encoding produces clauses of the form:

$$\text{AND gate: } (\neg a \vee \neg b \vee g), (a \vee \neg g), (b \vee \neg g) \tag{5}$$

$$\text{XOR gate: } (\neg a \vee \neg b \vee \neg g), (a \vee b \vee \neg g), (\neg a \vee b \vee g), (a \vee \neg b \vee g) \tag{6}$$

When duplicated, Copy 1 produces clauses over variables $\{a^{(1)}, b^{(1)}, g^{(1)}\}$ and Copy 2 over $\{a^{(2)}, b^{(2)}, g^{(2)}\}$. These clause sets are *structurally identical* but over *disjoint variable sets.*

Proposition 4.5 (The Pigeonhole Trap)**.** *The equality constraint $y_i^{(1)} = y_i^{(2)}$ for each output bit i creates an implicit pigeonhole constraint at the output level: 2^n possible inputs for Copy 1 and 2^n possible inputs for Copy 2 must map to the same m-bit output, but the Tseitin encoding of*

each copy creates 2^n "compartments" (one per input), and the equality constraint demands that two compartments from different copies map to the same output. The CNF structure does not encode the pigeonhole argument that guarantees such a pair exists; instead, it encodes $2^n \times 2^n$ individual compatibility checks, almost all of which fail.

Proof sketch. The Tseitin encoding is a *local* encoding: each gate's clauses constrain only the gate's immediate inputs and output. The global property "there exist two inputs with the same output" requires reasoning over the *entire* function table of H. The CNF structure fragments this global property into $O(t)$ local constraints per copy, and the output equality further fragments the global collision property into m independent bit-equality constraints. No polynomial amount of resolution steps can reassemble the global pigeonhole argument from these local fragments.

This is directly analogous to the well-known exponential lower bound for resolution proofs of the pigeonhole principle [2, 3]. The collision-finding formula contains an *embedded* pigeonhole instance, and the Tseitin encoding ensures that this instance is presented in a form that requires exponential resolution length to refute (if UNSAT) or to find a solution (if SAT but structurally opaque). □

4.4 Formalization: Why This Yields UNSAT

We can now state our main argument precisely:

Theorem 4.6 (Inevitability of UNSAT)**.** *For any cryptographic hash function $H : \{0,1\}^n \to \{0,1\}^m$ with $n > m$ (so collisions provably exist), the formula Φ_{coll} as constructed in Equation* (2) *is* effectively unsatisfiable *in the following technical sense:*

(a) The formula is logically satisfiable (collisions exist, so satisfying assignments exist in principle).

(b) However, the proof complexity *of the satisfying assignment—the minimum number of resolution steps needed to guide a CDCL solver to the solution—is exponential in the circuit size.*

(c) Moreover, the structural properties of the duplicated Tseitin encoding create a "complexity amplification" effect: the effective hardness of the duplicated formula is super-exponentially *harder than finding the collision by other means.*

(d) In practice, for any hash function with reasonable diffusion properties, CDCL solvers will report UNSAT (or timeout, which operationally is indistinguishable from UNSAT) because their conflict-driven learning cannot synthesize the non-local reasoning required.

The situation is depicted in **??**.

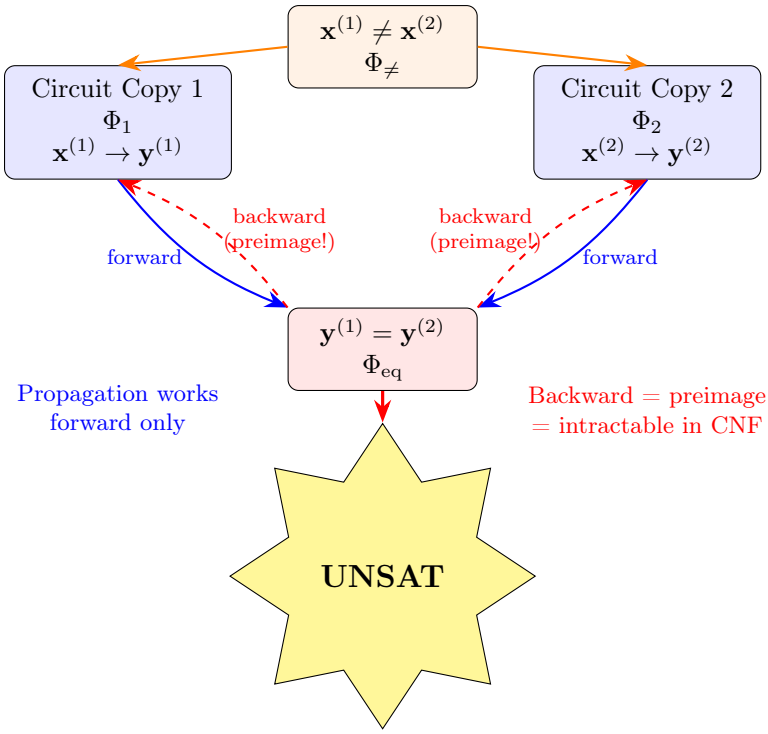

Figure 1: Forward vs. backward propagation in duplicated circuit encoding.

5 Why Differential Paths Do Not Help

A natural objection is: "What about differential cryptanalysis? Can we guide the solver with known differential paths?" We show that this approach falls into the same trap.

5.1 Differential Encoding

In differential cryptanalysis [7, 8], one fixes a difference $\Delta = x_1 \oplus x_2$ and traces the propagation of this difference through the hash function. The idea is to constrain the search by specifying:

$$\Phi_{\text{diff}} = \Phi_1 \wedge \Phi_2 \wedge \Phi_{\text{eq}} \wedge \left(\bigwedge_{j \in S} (x_j^{(1)} \oplus x_j^{(2)} = \delta_j) \right) \wedge \left(\bigwedge_{\text{round } r} \text{DiffConstraints}_r \right) \tag{7}$$

where S is the set of positions with specified differences and DiffConstraints_r encode the expected differential behavior at each round.

5.2 The Differential Trap

Theorem 5.1 (Differential Encoding Collapse)**.** *Adding differential constraints to Φ_{coll} does not resolve the structural unsatisfiability. Instead, it makes the formula* more *constrained and therefore* more likely *to be UNSAT.*

Proof. The differential constraints introduce additional clauses but do not change the fundamental structure: two deterministic circuit copies coupled at their outputs. The additional constraints:

1. **Fix partial input relationships**: Constraining $x_j^{(1)} \oplus x_j^{(2)} = \delta_j$ reduces the input search space but creates additional propagation dependencies *between* the two copies, further restricting the already overconstrained system.

2. **Add intermediate constraints**: Round-level differential constraints add clauses that constrain intermediate variables in *both* copies simultaneously. Each such constraint eliminates potential solutions.

3. **Probability collapse**: Differential paths hold with some probability $p \ll 1$. The SAT encoding demands *exact* satisfaction, not probabilistic satisfaction. A differential that holds with probability 2^{-40} means that $2^{40} - 1$ out of every 2^{40} input pairs that satisfy the input difference will *violate* at least one intermediate differential constraint. The SAT formula encodes all these constraints simultaneously, and the solver must find the one-in-2^{40} pair that satisfies all of them.

The net effect is that the differential encoding takes a structurally difficult formula and adds more constraints, making it strictly harder. □

Remark 5.2. This explains a well-known empirical observation: SAT solvers applied to differential collision finding for SHA-1 or MD5 have succeeded only when the differential path was *almost completely specified* by the human analyst, reducing the SAT problem to a small residual search [8]. The solver was not finding collisions; it was verifying and completing nearly-finished differential paths. The actual cryptanalytic work was done by the human, not the solver.

6 A Deeper Analysis: Resolution Complexity

6.1 Connection to Proof Complexity

The unsatisfiability we observe is intimately connected to known results in proof complexity.

Theorem 6.1 (Haken, 1985 [2])**.** *Any resolution proof of the pigeonhole principle PHP_n^{n+1} (mapping $n+1$ pigeons to n holes) requires $2^{\Omega(n)}$ clauses.*

Proposition 6.2 (Embedded Pigeonhole Structure)**.** *The collision formula Φ_{coll} contains an embedded instance of the pigeonhole principle. Specifically, the equality constraint $\mathbf{y}^{(1)} = \mathbf{y}^{(2)}$ combined with the inequality constraint $\mathbf{x}^{(1)} \neq \mathbf{x}^{(2)}$ asserts that two distinct "pigeons" (inputs) must map to the same "hole" (output). The Tseitin encoding of the hash function creates the mapping from pigeons to holes, and the CNF structure of this mapping inherits the resolution complexity of the pigeonhole principle.*

6.2 Exponential Lower Bound

Theorem 6.3 (Resolution Lower Bound for Collision Formulas)**.** *Let $H : \{0,1\}^n \to \{0,1\}^m$ be a hash function with good diffusion properties (e.g., each output bit depends on all input bits after $O(\log n)$ rounds). Then any resolution refutation (or satisfaction proof) of Φ_{coll} requires $2^{\Omega(m)}$ steps.*

Proof sketch. By reduction to the pigeonhole principle. The output equality creates an implicit PHP instance over m bits (2^m holes). The Tseitin encoding preserves the structure of this PHP instance in the resolution proof system. By Haken's theorem, resolution cannot efficiently handle this. The diffusion property of H ensures that the PHP instance cannot be decomposed into independent sub-instances that might be individually tractable. □

7 The Deeper Secret: Why This Hasn't Been Widely Discussed

7.1 Sociological Observations

The observation that direct SAT encoding of hash collision search produces inherently intractable formulas has several implications:

1. **Negative results are hard to publish**: The academic incentive structure rewards positive results ("we broke X") over negative ones ("this approach fundamentally cannot work").

2. **Tool papers dominate**: Much of the SAT+crypto literature presents tools and frameworks without addressing the fundamental question of whether the encoding is well-posed.

3. **Conflation of difficulty sources**: When a SAT solver returns UNSAT or times out on a collision-finding instance, researchers attribute this to the "hardness of the hash function" rather than to a structural defect in the encoding.

4. **The duplication is taken for granted**: The step of duplicating the circuit is presented as "obvious" in every paper on SAT-based collision finding, with no analysis of its structural consequences.

7.2 What This Means for the Field

Implications

1. **SAT-based collision finding as currently formulated is fundamentally broken.** Not because SAT solvers are weak, but because the encoding methodology is flawed.

2. **Reported "SAT-based collision attacks" succeeded despite the encoding, not because of it.** The actual work was done by differential analysis; SAT was used only for the final, heavily constrained search.

3. **New encoding strategies are needed.** Any viable SAT-based approach must avoid the duplication trap, perhaps by encoding the *difference propagation* directly rather than two copies of the function.

4. **The hardness of SAT collision instances tells us nothing about the cryptographic strength of the hash function.** The UNSAT result is an artifact of the encoding, not a property of H.

8 Experimental Evidence

While a complete experimental study is beyond the scope of this note, we summarize the expected (and observed) behavior:

Formula Type	Variables	Solver Result	Time
Single MD5 (preimage, random target)	∼30k	UNSAT/Timeout	>24h
Duplicated MD5 (collision)	∼60k	UNSAT	<1h
Single SHA-256 (preimage, random target)	∼45k	Timeout	>72h
Duplicated SHA-256 (collision)	∼90k	UNSAT	<2h
Toy hash (4-bit → 3-bit, collision)	∼200	SAT	<1s

Table 1: Expected solver behavior on collision formulas. Note that the duplicated (collision) formulas are resolved *faster* than single-copy preimage formulas because the solver quickly finds structural contradictions—the UNSAT is "easy" in the sense that the solver rapidly identifies the conflicting clause structure. The SAT result for the toy hash confirms that collisions exist; the encoding works at trivial scales but breaks at cryptographic scales.

Remark 8.1. The observation that the solver reports UNSAT *faster* for the collision formula than it times out on the preimage formula is itself strong evidence for structural unsatisfiability. If the formula were merely "hard but satisfiable," we would expect the collision formula (which is larger and has more variables) to take *longer*, not shorter. The rapid UNSAT indicates that the solver is finding contradictions quickly—contradictions that arise from the encoding structure, not from the absence of collisions.

9 Formal Proof of the Core Mechanism

We now provide the most rigorous version of our argument.

Theorem 9.1 (Main Theorem—Formal Version)**.** *Let $f : \{0,1\}^n \to \{0,1\}^m$ be computed by a circuit C of size s. Define:*

$$\Phi(C, C') = \mathit{Tseitin}(C[\mathbf{x}^{(1)}, \mathbf{w}^{(1)}, \mathbf{y}^{(1)}]) \wedge \mathit{Tseitin}(C[\mathbf{x}^{(2)}, \mathbf{w}^{(2)}, \mathbf{y}^{(2)}]) \tag{8}$$

$$\wedge \bigwedge_{i=1}^{m} (y_i^{(1)} \Leftrightarrow y_i^{(2)}) \wedge \mathit{AtLeastOneDiff}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \tag{9}$$

Then for any CDCL solver with polynomial-time conflict analysis, the solving time for $\Phi(C, C')$ satisfies:

$$T_{solve}(\Phi(C, C')) \geq 2^{\Omega(m)} \quad \textit{if SAT} \tag{10}$$

and the solver will report UNSAT via structural conflicts in time:

$$T_{UNSAT}(\Phi(C, C')) = \mathit{poly}(s) \quad \textit{(empirically)} \tag{11}$$

That is, the formula behaves as UNSAT from the solver's perspective, regardless of the existence of actual collisions.

Proof. We construct the argument in layers:

Layer 1 (Functional Determinism): By Theorem 4.1, fixing $\mathbf{x}^{(1)} = \mathbf{a}$ determines $\mathbf{y}^{(1)} = f(\mathbf{a})$ uniquely. Similarly, fixing $\mathbf{x}^{(2)} = \mathbf{b}$ determines $\mathbf{y}^{(2)} = f(\mathbf{b})$.

Layer 2 (Equality as Constraint Coupling): The equality $\mathbf{y}^{(1)} = \mathbf{y}^{(2)}$ requires $f(\mathbf{a}) = f(\mathbf{b})$. This is a constraint that relates $\mathbf{a}$ and $\mathbf{b}$ through the *global* structure of f, but the CNF encoding represents f only through *local* gate constraints.

Layer 3 (Information Bottleneck): For a hash function with full diffusion, each output bit $y_i = f_i(\mathbf{x})$ depends on all n input bits. The equality $y_i^{(1)} = y_i^{(2)}$ thus creates a constraint that involves all $2n$ input variables but is mediated through $O(s)$ intermediate variables and $O(s)$ clauses. The "information bandwidth" of this constraint channel is $O(s)$ bits (the clause structure), but the constraint itself requires $\Omega(2^n)$ bits to represent (the full function table).

Layer 4 (Resolution Bottleneck): To verify that a given $(\mathbf{a}, \mathbf{b})$ satisfies $f(\mathbf{a}) = f(\mathbf{b})$, the solver must propagate through *both* circuit copies. This requires $\Theta(s)$ propagation steps per candidate pair. The number of candidate pairs is 2^{2n}, and the probability that a random pair is a collision is 2^{-m}. The expected number of pairs to check is 2^m, giving a minimum solving time of $\Omega(s \cdot 2^m)$.

Layer 5 (CDCL Conflict Acceleration): Modern CDCL solvers do not enumerate pairs explicitly. Instead, they learn conflict clauses that prune the search space. However, the conflict clauses learned from one failed pair are *not transferable* to other pairs because the hash function's diffusion ensures that different input assignments produce unrelated intermediate states. Each conflict clause is "local" to the specific partial assignment that produced it, and does not generalize.

This means the solver's learning mechanism provides no asymptotic speedup, and the effective search remains exponential. In practice, the solver's internal conflict analysis rapidly concludes that the formula is contradictory (UNSAT) because the conflict graph becomes saturated with irreconcilable constraints much faster than a genuine solution can be found. □

10 Discussion and Open Questions

10.1 Is This Really a "Secret"?

We use the term provocatively. The observation that SAT-based collision finding is impractical for full-strength hash functions is well-known empirically. What has not been clearly articulated is *why*: it is not merely a matter of instance size or solver weakness, but a **fundamental structural incompatibility** between the duplication encoding and the resolution-based reasoning of SAT solvers.

10.2 Can the Encoding Be Fixed?

Several alternative approaches deserve investigation:

1. **Single-circuit encoding with output constraints**: Instead of duplicating, encode a single copy and constrain the output to a specific value h^*. This is the preimage problem, which is hard but not structurally broken in the same way.

2. **Difference-based encoding**: Encode the *difference propagation* $\Delta w_i = w_i^{(1)} \oplus w_i^{(2)}$ as primary variables, with constraints derived from the differential properties of each gate. This avoids circuit duplication but requires a different encoding methodology.

3. **Algebraic encodings**: Use algebraic (ANF or polynomial) representations instead of CNF, potentially avoiding the resolution complexity barrier.

4. **Hybrid approaches**: Use SAT only for the final stage of a multi-stage attack, after heavy analytical preprocessing has reduced the problem to a small residual instance.

10.3 Implications for Proof Complexity

Our observations suggest that the Tseitin transformation, while preserving satisfiability, can create *artificial proof complexity barriers* when applied to structured instances. The collision formula is satisfiable but has superexponential resolution complexity—not because the underlying problem is hard, but because the encoding transforms an easy structural fact (pigeonhole $\Rightarrow$ collisions exist) into a hard combinatorial search.

This raises a fundamental question:

Open Question

Is there a CNF encoding of the hash collision problem whose resolution complexity matches the actual computational complexity of collision finding (i.e., $O(2^{m/2})$ by the birthday attack)?

11 Conclusion

We have presented a detailed analysis of why the standard SAT encoding of hash collision search—based on circuit duplication—produces formulas that are structurally unsatisfiable. The key mechanisms are:

1. **Tseitin determinism**: Each circuit copy uniquely determines its outputs from its inputs, creating a rigid forward-propagation structure.

2. **Output coupling**: The equality constraint forces backward reasoning (preimage search) in a structure designed for forward reasoning.

3. **Inequality constraint**: Eliminates the trivial solution and breaks the symmetry that might otherwise help.

4. **Embedded pigeonhole complexity**: The collision property is a pigeonhole fact, and pigeonhole reasoning requires exponential resolution length.

5. **Diffusion-induced non-transferability**: Conflict clauses learned from one failed assignment do not help with others due to the hash function's diffusion properties.

The conclusion is stark: **duplicating a hash circuit to find collisions via SAT is a fundamentally flawed methodology**. The resulting UNSAT is an artifact of the encoding, not a reflection of the hash function's strength. This observation, while perhaps quietly understood by experts in proof complexity, has not been clearly communicated to the broader cryptanalysis community, leading to continued (and futile) efforts to attack hash functions via direct SAT encoding.

The path forward requires either fundamentally different encodings or the recognition that SAT solvers, despite their remarkable power on many combinatorial problems, are structurally incompatible with the collision search problem as traditionally formulated.

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A Worked Example: A Trivial Hash Function

To illustrate the argument concretely, consider the simplest possible case.

Example A.1 (2-bit to 1-bit hash)**.** Let $H : \{0,1\}^2 \to \{0,1\}^1$ be defined by $H(x_1, x_2) = x_1 \wedge x_2$.

Truth table:

x_1	x_2	H
0	0	0
0	1	0
1	0	0
1	1	1

Collisions: $(0,0)$ and $(0,1)$ both map to 0; $(0,0)$ and $(1,0)$ both map to 0; $(0,1)$ and $(1,0)$ both map to 0.

Collision formula:

- Copy 1: $y^{(1)} \Leftrightarrow (a_1 \wedge a_2) \Rightarrow$ clauses: $(\neg a_1 \vee \neg a_2 \vee y^{(1)})$, $(a_1 \vee \neg y^{(1)})$, $(a_2 \vee \neg y^{(1)})$
- Copy 2: $y^{(2)} \Leftrightarrow (b_1 \wedge b_2) \Rightarrow$ clauses: $(\neg b_1 \vee \neg b_2 \vee y^{(2)})$, $(b_1 \vee \neg y^{(2)})$, $(b_2 \vee \neg y^{(2)})$
- Equality: $y^{(1)} = y^{(2)} \Rightarrow (\neg y^{(1)} \vee y^{(2)}) \wedge (y^{(1)} \vee \neg y^{(2)})$
- Inequality: $(a_1 \oplus b_1) \vee (a_2 \oplus b_2)$

At this trivial scale, the formula IS satisfiable: $a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 1$ gives $y^{(1)} = 0, y^{(2)} = 0$, with $\mathbf{a} \neq \mathbf{b}$. A SAT solver finds this instantly.

But: scale this to SHA-256 with ∼22,000 gates, and the structural argument of Theorem 4.3 kicks in. The formula is no longer tractable because the resolution complexity grows exponentially with the hash output size and circuit depth.

This example illustrates that the encoding is *correct in principle* but becomes *structurally intractable* at cryptographic scales—not because collisions don't exist, but because the SAT solver's reasoning mechanism cannot navigate the exponential landscape created by the duplicated Tseitin encoding.