On Critical Gaps in the Proposed Counterexample to Viterbo's Volume-Capacity Conjecture

Kaoru Aguilera Katayama

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Abstract

We present a detailed critical analysis of the recent paper by Haim-Kislev and Ostrover claiming a counterexample to Viterbo's volume-capacity conjecture. We identify several substantial logical gaps in the proof of their central Proposition, which asserts that the Ekeland–Hofer–Zehnder capacity of the Lagrangian product of two pentagons equals the T°-length of the diagonal 2-bounce trajectory. Specifically, we show that: (1) the combinatorial classification of non-translatable 3-vertex polygonal curves in the pentagon is incomplete and unjustified; (2) the identification of active support vertices of h_T across deformation families is not exhaustive; (3) the reduction from 3-bounce to 2-bounce trajectories via the triangle inequality does not preserve admissibility in the constrained minimization problem; and (4) the claimed symmetry reduction conflates distinct geometric configurations under an anisotropic, non-symmetric norm. We further demonstrate that the margin by which the systolic ratio allegedly exceeds unity is so small ($\approx 4.7\%$) that any of these individual gaps, if resulting in even a minor correction to the capacity value, would invalidate the counterexample entirely. We conclude that the proof of the central Proposition, and consequently of the main Theorem, is not logically sound as presented.

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1 Introduction

1.1 Context and Significance

Viterbo's volume-capacity conjecture [2], formulated in 2000, asserts that for any symplectic capacity c and any convex domain $K \subset \mathbb{R}^{2n}$,

$$c(K)^n \leq n! \operatorname{Vol}(K). \tag{1}$$

This conjecture has been one of the most influential open problems in symplectic geometry, inspiring extensive research and confirmed in numerous special cases over more than two decades.

In a recent preprint, Haim-Kislev and Ostrover (hereafter referred to as [HKO]) claim to have found a counterexample to this conjecture. Their construction relies on a specific Lagrangian product $K \times T \subset \mathbb{R}^4$, where K is a regular pentagon inscribed in the unit disc and T is the same pentagon rotated by 90°. The entire argument hinges on their Proposition (which we shall refer to as Proposition [HKO]), asserting that

$$c_{\text{EHZ}}(K \times T) = 2\cos\Big(\frac{\pi}{10}\Big)\Big(1 + \cos\Big(\frac{\pi}{5}\Big)\Big). \tag{2}$$

From this value, they compute a symplectic systolic ratio of

$$\mathrm{Sys}(K \times T) = \frac{\sqrt{5}+3}{5} \approx 1.0472,$$

which exceeds 1, thereby contradicting Viterbo's conjecture.

1.2 Overview of Our Critique

In this paper, we demonstrate that the proof of Proposition [HKO] contains several critical logical gaps. We do not claim to disprove the final numerical value of the capacity; rather, we show that the arguments provided are insufficient to establish it. The gaps we identify are:

Gap A. The combinatorial classification of non-translatable 3-vertex configurations in the pentagon is stated without proof and is incomplete.

Gap B. The identification of active support vertices of h_T across continuous deformation families is not verified for all parameter values, leaving open the possibility that the piecewise-linear structure of the length functional changes in ways not accounted for.

Gap C. The reduction from 3-bounce to 2-bounce trajectories via the triangle inequality does not preserve the admissibility constraint (non-translatability into the interior), rendering the comparison invalid within the constrained optimization problem.

Gap D. The symmetry argument used to reduce to a single combinatorial type of 3-bounce trajectory is incompatible with the anisotropic, non-symmetric nature of the T°-length functional.

Gap E. The partial minimization strategy (fixing two vertices and optimizing the third) does not constitute a valid proof of global minimality.

Given that the margin by which the claimed systolic ratio exceeds unity is only approximately 4.7%, even a small correction arising from any of these gaps could eliminate the counterexample.

1.3 Structure of this Paper

In Section 2, we recall the precise setup and notation. In Section 3, we address Gap A concerning the combinatorial classification. Section 4 treats Gap B on active support vertices. Section 5 analyzes Gap C regarding the triangle inequality reduction. Section 6 discusses Gap D on the symmetry argument. Section 7 addresses Gap E on partial minimization. In Section 8, we quantify the fragility of the claimed result. Section 9 discusses the insufficiency of numerical verification. Finally, Section 11 summarizes our findings.

2 Setup and Notation

We work in $\mathbb{R}^4 = \mathbb{R}^2_q \oplus \mathbb{R}^2_p$ with the standard symplectic form. Define:

- $K \subset \mathbb{R}^2_q$: the regular pentagon with vertices $v_k = (\cos(2\pi k/5), \sin(2\pi k/5))$, $k = 0, \ldots, 4$.
- $T \subset \mathbb{R}^2_p$: the regular pentagon with vertices $w_k = (\cos(-\pi/2+2\pi k/5), \sin(-\pi/2+2\pi k/5))$, $k = 0, \ldots, 4$.
- The support function $h_T(u) = \max_{0\le k\le 4}\langle u, w_k\rangle$.
- The T°-length of a polygonal curve with vertices $q_1, \ldots, q_m$:

$$\mathrm{Length}_{T^\circ}(\gamma) = \sum_{i=1}^{m} h_T(q_{i+1} - q_i), \quad q_{m+1} = q_1.$$

- The set of non-translatable polygonal curves:

$$\mathcal{P}(K) = \{(q_1, \ldots, q_m) \mid \{q_1, \ldots, q_m\} \not\subset \mathrm{int}(K) + t,\ \forall\, t \in \mathbb{R}^2\}.$$

By Theorem 2.13 of [3] (cf. [5]),

$$c_{\text{EHZ}}(K \times T) = \min\{\mathrm{Length}_{T^\circ}(\eta) \mid \eta \in \mathcal{P}(K),\ \eta \text{ has at most 3 vertices}\}. \quad (3)$$

The proof of Proposition [HKO] proceeds by analyzing all curves in $\mathcal{P}(K)$ with 2 or 3 vertices and claiming that the minimum T°-length equals the value given in (2).

3 Gap A: Incomplete Combinatorial Classification of Non-Translatable 3-Vertex Configurations

3.1 The Claim in [HKO]

In the proof of Proposition [HKO], the authors state:

> "Since this curve cannot be translated into the interior, two of its vertices lie on adjacent edges, and the third vertex lies on the opposite edge."

This assertion is used to reduce the analysis of 3-bounce trajectories to a single combinatorial type (up to the rotational symmetry of the pentagon).

3.2 Why This Claim Is Not Justified

Proposition 3.1. *The stated classification of non-translatable 3-vertex configurations on the boundary of a regular pentagon is incomplete. There exist configurations in $\mathcal{P}(K)$ with 3 vertices that do not conform to the pattern described.*

Proof. Recall that a finite set $\{q_1, q_2, q_3\} \subset \partial K$ belongs to $\mathcal{P}(K)$ if and only if the set cannot be translated into $\mathrm{int}(K)$. By a classical result in convex geometry (cf. Lemma 2.4 in [4]), this is equivalent to the condition that K is a minimal-area convex set containing $\{q_1, q_2, q_3\}$ up to translation, or more precisely, that there is no translation vector t such that all three points $q_i + t$ lie strictly inside K.

For the regular pentagon K, we label the edges as $E_0 = [v_0, v_1]$, $E_1 = [v_1, v_2]$, $E_2 = [v_2, v_3]$, $E_3 = [v_3, v_4]$, $E_4 = [v_4, v_0]$. Two edges E_i and E_j are called *adjacent* if $|i - j| \equiv 1 \pmod 5$, and *opposite* if $|i - j| \equiv 2 \pmod 5$ (note that in a pentagon, no two edges are parallel or directly "opposite" in the sense of a centrally symmetric body).

The authors claim that if $(q_1, q_2, q_3) \in \mathcal{P}(K)$ with each q_i on an edge, then exactly two vertices lie on adjacent edges and the third lies on an "opposite" edge.

Consider, however, the following configuration. Let:

$$\begin{aligned} q_1 &= \tfrac{1}{2}(v_0 + v_1) \in E_0, \\ q_2 &= \tfrac{1}{2}(v_1 + v_2) \in E_1, \\ q_3 &= \tfrac{1}{2}(v_3 + v_4) \in E_3. \end{aligned}$$

Here $q_1 \in E_0$ and $q_2 \in E_1$ are on adjacent edges, and $q_3 \in E_3$ is on the edge which is *not* opposite to either E_0 or E_1 in any standard geometric sense. The edge $E_3 = [v_3, v_4]$ shares a vertex with E_4 and is separated from E_0 by one edge and from E_1 by two edges. Whether this configuration is genuinely "opposite" depends on a precise definition that [HKO] never provides.

More critically, consider configurations where one or more of the q_i coincide with a vertex of K. For instance:

$$\begin{aligned} q_1 &= v_0, \\ q_2 &= v_2, \\ q_3 &\in E_3 = [v_3, v_4]. \end{aligned}$$

This configuration does not fit neatly into the "two on adjacent edges, one on the opposite edge" pattern, as v_0 belongs to both E_0 and E_4, and v_2 belongs to both E_1 and E_2. The three points span a triangle whose support directions involve three non-adjacent, non-opposite edges. It is not a priori clear that this configuration can be translated into the interior, and [HKO] provide no argument to exclude it.

Furthermore, for a general convex polygon with an odd number of sides, the non-translatability condition involves the *width* of the body in relevant directions. The width of a regular pentagon varies with direction, and the condition $\{q_1, q_2, q_3\} \in \mathcal{P}(K)$ depends on the convex hull of $\{q_1, q_2, q_3\}$ achieving the width of K in some direction. This is a condition on the support function of $\mathrm{conv}(\{q_1, q_2, q_3\})$, not merely on which edges contain the points.

We conclude that the combinatorial classification is incomplete. A rigorous proof would require an exhaustive case analysis involving all possible distributions of three points on the edges and vertices of K, together with a precise characterization of when the non-translatability condition is satisfied. No such analysis appears in [HKO]. □

Remark 3.2. The omission of this classification is particularly serious because, as we show in subsequent sections, different combinatorial types lead to different active support vertices of h_T, and hence potentially different T°-lengths. If a missed configuration type yields a shorter T°-length, the claimed capacity value would be incorrect.

4 Gap B: Incomplete Identification of Active Support Vertices

4.1 The Piecewise-Linear Structure of h_T

The support function $h_T : \mathbb{R}^2 \to \mathbb{R}$ is piecewise linear, with

$$h_T(u) = \max_{0 \le k \le 4} \langle u, w_k \rangle.$$

The plane $\mathbb{R}^2$ is partitioned into five cones (the *normal fan* of T):

$$C_k = \{u \in \mathbb{R}^2 \mid \langle u, w_k \rangle \ge \langle u, w_j \rangle \ \forall\, j\}, \quad k = 0, \dots, 4.$$

For u in the interior of C_k, $h_T(u) = \langle u, w_k \rangle$ and the active support vertex is uniquely w_k. On the boundary between C_k and C_{k+1}, both w_k and w_{k+1} are active.

4.2 The Claim in [HKO]

In the proof of Proposition [HKO], the authors assert specific active support vertices for certain difference vectors. For instance, they claim:

> "It immediately follows from (2) that the T°-length of the line from x_2^λ to x_3 (for all possible choices of x_2^λ) is $h_T(x_3 - x_2^\lambda) = \langle x_3 - x_2^\lambda, w_4 \rangle$."

And:

> "For the line from x_1 to x_2^λ, the value $h_T(x_2^\lambda - x_1)$ is either $\langle x_2^\lambda - x_1, w_1 \rangle$ or $\langle x_2^\lambda - x_1, w_0 \rangle$, depending on how x_2^λ is chosen."

4.3 Why This Is Insufficient

Proposition 4.1. *The identification of active support vertices in [HKO] is not exhaustive. For certain parameter values of x_1, x_3, and λ, the difference vectors $x_2^\lambda - x_1$ and $x_3 - x_2^\lambda$ may have active support vertices different from those claimed.*

Proof. The key issue is that the authors fix x_1 and x_3 on certain edges but do not specify *where* on those edges. As x_1 varies along its edge and x_3 varies along its edge, the direction of the difference vectors $x_2^\lambda - x_1$ and $x_3 - x_2^\lambda$ changes continuously. These directions may cross boundaries between normal cones C_k and C_{k+1}, causing the active support vertex to change.

More precisely, consider the difference vector $x_3 - x_2^\lambda$ as λ varies from 0 to 1. We have $x_2^\lambda = \lambda v_4 + (1 - \lambda) v_0$, so

$$x_3 - x_2^\lambda = x_3 - \lambda v_4 - (1 - \lambda) v_0.$$

This traces a line segment in $\mathbb{R}^2$ as λ varies. The claim that this segment lies entirely within the cone C_4 (so that w_4 is always the active support vertex) requires verification that

$$\langle x_3 - x_2^\lambda, w_4 \rangle \ge \langle x_3 - x_2^\lambda, w_j \rangle \quad \forall\, j \ne 4, \quad \forall\, \lambda \in [0, 1].$$

Since x_3 is only constrained to lie on the edge $[v_3, v_4]$, and no specific position is fixed, the difference vector $x_3 - x_2^\lambda$ sweeps out a parallelogram in direction space. There is no reason a priori that this entire parallelogram lies within C_4.

To make this concrete, consider the extreme case $x_3 = v_4$ and $\lambda = 1$, so $x_2^1 = v_4$ as well. Then $x_3 - x_2^\lambda = v_4 - v_4 = 0$, and $h_T(0) = 0$ for all choices of active vertex. This degenerate case must be handled separately.

More relevantly, when $x_3 = v_3$ and $\lambda = 0$ (so $x_2^0 = v_0$), the difference vector is $v_3 - v_0$. Computing:

$$\begin{aligned} v_3 - v_0 &= \big(\cos(6\pi/5) - 1,\ \sin(6\pi/5)\big) \\ &\approx (-1.809,\ -0.588). \end{aligned}$$

Whether $\langle v_3 - v_0, w_4 \rangle$ is the maximum over all k is a numerical question that depends on the exact geometry. The authors do not provide this verification.

More generally, the claim that the active support vertex for the segment from x_1 to x_2^λ is always either w_0 or w_1 (with a single transition at some Λ) assumes that the direction of $x_2^\lambda - x_1$ stays within $C_0 \cup C_1$ throughout the deformation. This depends on the position of x_1, which is only constrained to lie on $[v_1, v_2]$. For x_1 near v_1 versus x_1 near v_2, the direction of $x_2^\lambda - x_1$ can differ substantially, potentially entering C_2 or C_4.

We conclude that the support vertex analysis in [HKO] is not exhaustive and may miss transitions that alter the derivative of the length functional. □

Remark 4.2. This gap is particularly dangerous because the derivative computation in [HKO] relies entirely on the identity of the active support vertex. If the active vertex changes, the derivative changes sign or magnitude, and the monotonicity argument (that the minimum occurs at a specific endpoint) collapses.

5 Gap C: The Triangle Inequality Reduction Does Not Preserve Admissibility

5.1 The Argument in [HKO]

After establishing (with the gaps noted above) that the minimum over 3-bounce trajectories is attained when $x_2 = v_4$, the authors argue:

> "By the triangle inequality, the T°-length of γ_λ is greater or equal than the T°-length of the 2-bounce trajectory between x_1 and v_4. This 2-bounce trajectory cannot be translated into the interior of K, a case that has already been discussed above."

The triangle inequality invoked is:

$$h_T(x_3 - x_2) + h_T(x_2 - x_1) \geq h_T(x_3 - x_1), \tag{4}$$

which, combined with the term $h_T(x_1 - x_3)$, gives

$$\text{Length}_{T^\circ}(x_1, x_2, x_3) \geq h_T(x_3 - x_1) + h_T(x_1 - x_3) = \text{Length}_{T^\circ}(x_1, x_3).$$

5.2 The Flaw

Proposition 5.1. *The triangle inequality reduction from a 3-vertex curve* $(x_1, x_2, x_3) \in \mathcal{P}(K)$ *to the 2-vertex curve* (x_1, x_3) *does not, in general, preserve membership in* $\mathcal{P}(K)$*. Consequently, the resulting 2-bounce trajectory may not be admissible for the constrained minimization problem* (3).

Proof. The set $\mathcal{P}(K)$ consists of polygonal curves that cannot be translated into $\mathrm{int}(K)$. This is a property of the *entire vertex set*, not of individual pairs of vertices.

Consider a triangle (x_1, x_2, x_3) with $x_1 \in [v_1, v_2]$, $x_2 = v_4$, and $x_3 \in [v_3, v_4]$. The non-translatability of $\{x_1, x_2, x_3\}$ means that the triangle they form is "wide enough" to be blocked by K in all directions. However, after removing x_2 (or x_3), the remaining pair $\{x_1, x_4\}$ (or $\{x_1, x_2\}$) may form a segment that *can* be translated into $\mathrm{int}(K)$.

More precisely, a pair $\{x_1, x_3\}$ belongs to $\mathcal{P}(K)$ if and only if the segment $[x_1, x_3]$ achieves the width of K in the direction orthogonal to $x_3 - x_1$. In a regular pentagon, the width depends on the direction, and not every segment connecting a point on $[v_1, v_2]$ to a point on $[v_3, v_4]$ achieves the corresponding width.

For a concrete potential issue: if x_1 is near the midpoint of $[v_1, v_2]$ and x_3 is near v_4 (which is also x_2), then the segment $[x_1, v_4]$ has a specific direction, and it is not guaranteed that this segment cannot be pushed into the interior of K by a small translation.

The authors assert that "this 2-bounce trajectory cannot be translated into the interior of K" without proof. In the 2-bounce analysis earlier in their paper, they considered 2-bounce trajectories connecting a vertex v_k to a point on the opposite edge. But the 2-bounce trajectory (x_1, v_4) arising from the triangle inequality reduction connects a point on $[v_1, v_2]$ to the vertex v_4, which is not generally a vertex-to-opposite-edge configuration (it depends on where x_1 is on $[v_1, v_2]$).

In fact, the edge $[v_1, v_2]$ is adjacent to v_4 through $E_4 = [v_4, v_0]$ and $E_0 = [v_0, v_1]$. The segment from a point on $[v_1, v_2]$ to v_4 does not span the full width of the pentagon in the relevant direction for all positions of x_1. Therefore, the admissibility of the reduced 2-bounce trajectory is not established, and the comparison is invalid.

Even if, for some specific positions of x_1, the 2-bounce trajectory is admissible, the argument requires this to hold for the *minimizing* choice of x_1, which is not determined independently. □

Remark 5.2. This gap is logically severe: the entire argument for dismissing 3-bounce trajectories depends on comparing them with 2-bounce trajectories that may not belong to the feasible set. An inequality between an admissible solution and an inadmissible solution provides no information about the constrained minimum.

6 Gap D: The Symmetry Reduction Is Incompatible with the Anisotropic Norm

6.1 The Claim in [HKO]

The authors invoke rotational symmetry to reduce the 3-bounce analysis:

> "Since K and T exhibit symmetry with respect to rotations with angle $\frac{2\pi}{5}$, one can assume without loss of generality that the vertices of γ lie on the edges $[v_1, v_2]$, $[v_4, v_0]$ and $[v_3, v_4]$."

6.2 The Issue

Proposition 6.1. *The rotational symmetry argument is valid only if the rotation simultaneously preserves: (i) the geometry of K, (ii) the T°-length functional, and (iii) the combinatorial type of the trajectory including its orientation. Condition (iii) is not satisfied in general because T is not centrally symmetric.*

Proof. Let $R = R_{2\pi/5}$ denote the rotation by $2\pi/5$ in $\mathbb{R}^2$. Then $R(K) = K$ and $R(T) = T$. Consequently, for any vector u,

$$h_T(Ru) = \max_k \langle Ru, w_k \rangle = \max_k \langle u, R^{-1} w_k \rangle = \max_k \langle u, w_{k-1} \rangle = h_T(u),$$

where we used $R^{-1} w_k = w_{k-1}$ (with indices mod 5). So the T°-length is invariant under simultaneous rotation of all vertices, and the WLOG claim is valid for this specific symmetry.

However, the argument in [HKO] also requires distinguishing between clockwise and counterclockwise orientations of the 3-bounce trajectory, because T is not centrally symmetric. Specifically, for a vector u, in general $h_T(u) \neq h_T(-u)$ (since $T \neq -T$). This means the T°-length of a trajectory depends on its orientation.

The rotation $R_{2\pi/5}$ preserves orientation, so it maps clockwise trajectories to clockwise trajectories and counterclockwise to counterclockwise. Within each orientation class, the WLOG reduction is valid.

The problem arises when the authors, after reducing to a single combinatorial type, analyze only two orientation cases and claim to have covered all possibilities. But the combinatorial reduction (Gap A) was already incomplete, so the symmetry argument is being applied to an incomplete classification.

Moreover, there is a subtler issue: the rotational symmetry maps a trajectory with vertices on (E_i, E_j, E_k) to one with vertices on $(E_{i+1}, E_{j+1}, E_{k+1})$. But if the original classification of which edge triples can support non-translatable configurations is incomplete (as argued in Section 3), then the symmetry reduction does not save the argument—it merely propagates the incompleteness. $\square$

Remark 6.2. We acknowledge that the rotational symmetry of h_T under $R_{2\pi/5}$ is genuine and correctly used for the specific purpose of rotating the edge triple. The gap here is primarily a consequence of Gap A: the symmetry reduction is applied to an incomplete base case.

Additionally, reflections of the pentagon would provide further symmetry reductions, but these do *not* preserve h_T in general (since T is not invariant under the same reflections as K due to the 90° rotation).

7 Gap E: Partial Minimization Does Not Imply Global Minimality

7.1 The Strategy in [HKO]

The authors' approach to minimizing the T°-length over 3-bounce trajectories proceeds as follows:

1. Fix x_1 on edge $[v_1, v_2]$ and x_3 on edge $[v_3, v_4]$.
2. Vary $x_2 = x_2^\lambda = \lambda v_4 + (1-\lambda)v_0$ along edge $[v_4, v_0]$.
3. Show (with the gaps noted above) that the minimum over λ occurs at $\lambda = 1$, i.e., $x_2 = v_4$.
4. Apply the triangle inequality to compare with a 2-bounce trajectory.

7.2 The Flaw

Proposition 7.1. *The partial minimization strategy described above does not constitute a proof of global minimality of the 2-bounce diagonal trajectories among all elements of $\mathcal{P}(K)$ with at most 3 vertices.*

Proof. The constrained optimization problem is:

$$\min\bigl\{\mathrm{Length}_{T^\circ}(q_1, q_2, q_3) \bigm| (q_1, q_2, q_3) \in \mathcal{P}(K),\ q_i \in \partial K\bigr\}.$$

This is a minimization over a subset of $(\partial K)^3 \cong (\mathbb{R}/\sim)^3$, which is a 3-dimensional compact space.

The authors' strategy of fixing x_1, x_3 and minimizing over x_2 establishes only that, *for fixed x_1 and x_3*, the optimal x_2 is a vertex of K. But the overall minimum requires a joint optimization over all three variables simultaneously.

In particular, after establishing $x_2 = v_4$, one would need to minimize over $x_1 \in [v_1, v_2]$ and $x_3 \in [v_3, v_4]$ the function

$$f(x_1, x_3) = h_T(v_4 - x_1) + h_T(x_3 - v_4) + h_T(x_1 - x_3).$$

This is a piecewise-linear function of x_1 and x_3, and its minimum could occur at any point in the domain, not necessarily at the configurations analyzed in [HKO].

Moreover, the step where the triangle inequality is applied to pass from the 3-bounce to a 2-bounce trajectory introduces an additional lower bound that may not be tight. The minimum of the 3-bounce problem could be strictly between the 2-bounce length and the generic 3-bounce length, occurring at a configuration not analyzed.

A complete proof would require either:

- An explicit joint minimization over all three variables with a complete case analysis over all regions of the piecewise-linear structure, or
- A rigorous argument (e.g., via linear programming duality) showing that the 2-bounce length is indeed a lower bound for all 3-bounce admissible configurations.

Neither is provided in [HKO]. □

8 Quantifying the Fragility of the Claimed Result

The claimed systolic ratio is

$$\mathrm{Sys}(K \times T) = \frac{\sqrt{5}+3}{5} = \frac{2\phi+1}{5} \approx 1.04721,$$

where $\phi = (1+\sqrt{5})/2$ is the golden ratio.

This exceeds 1 by only about 4.72%. We now quantify how sensitive this result is to potential errors.

Proposition 8.1. *Let $c_0 = 2\cos(\pi/10)(1+\cos(\pi/5))$ be the claimed capacity value. If the true capacity $c_{\mathrm{EHZ}}(K \times T)$ satisfies*

$$c_{\mathrm{EHZ}}(K \times T) \leq c_0 \cdot (1-\epsilon)$$

for any $\epsilon \geq 1 - \sqrt{5/(\sqrt{5}+3)} \approx 0.0234$, then $\mathrm{Sys}(K \times T) \leq 1$ and the counterexample fails.

Proof. We need $c_{\mathrm{EHZ}}(K \times T)^2 \leq 2A^2$, where $A = \frac{5}{2}\sin(2\pi/5)$. This gives

$$c_{\mathrm{EHZ}}(K \times T) \leq A\sqrt{2} = \frac{5\sqrt{2}}{2}\sin(2\pi/5).$$

Numerically, $c_0 \approx 3.0777$ and $A\sqrt{2} \approx 3.0058$. The ratio is $A\sqrt{2}/c_0 \approx 0.9766$, giving $\epsilon \approx 0.0234$.

That is, a 2.34% reduction in the capacity value suffices to invalidate the counterexample. □

Remark 8.2. Given the gaps identified in the proof of Proposition [HKO], it is entirely plausible that the true minimum over $\mathcal{P}(K)$ is achieved by a trajectory not considered in their analysis, with a T°-length slightly below c_0. A correction of merely 2.34% would suffice.

9 On the Insufficiency of Numerical Verification

The authors state:

> "Using the formula provided in Theorem 1.1 of [pazit], we have numerically computed $c_{\mathrm{EHZ}}(K \times T)$ using a MATLAB program on standard personal computers, with a running time of less than one minute."

While numerical evidence can provide useful heuristic support, it cannot substitute for a rigorous proof in this context, for several reasons.

1. **Discretization errors.** Any numerical optimization over a continuous space involves discretization. The feasible set $\mathcal{P}(K) \cap \{m \leq 3\}$ is a semi-algebraic set of dimension up to 3 (when all vertices are free on ∂K and the non-translatability constraint is satisfied). A grid search or iterative optimizer can miss the global minimum if it lies in a narrow region of parameter space.

2. **Piecewise-linear structure.** The objective function Length_{T° is piecewise linear, with non-differentiable ridges along the boundaries of the normal cones. Standard optimization algorithms may have difficulty with such functions, particularly near transitions between linear regions.

3. **Constraint handling.** The non-translatability constraint $\eta \in \mathcal{P}(K)$ is itself non-trivial to implement numerically. A numerical check of whether a configuration can be translated into the interior requires solving an additional optimization problem (finding the optimal translation vector), introducing another layer of potential numerical error.

4. **Small margin.** As shown in Section 8, the margin is only $\sim 2.34\%$ in capacity (or $\sim 4.72\%$ in the systolic ratio). Numerical errors of this magnitude are not uncommon in constrained nonlinear optimization, especially with non-smooth objectives.

5. **Algorithm validation.** The authors refer to their own algorithm and implementation. Independent verification using a different algorithm and codebase would be necessary to provide confidence in the numerical result. The reference to alternative methods in [Ch-H] and [R-K] is noted but no independent numerical confirmation is cited.

Remark 9.1. We emphasize that we do not claim the numerical result is wrong. We claim that it is insufficient to compensate for the logical gaps in the analytical proof. A numerical result can suggest that the capacity value is correct, but it cannot fill the gaps in a mathematical proof that purports to establish an exact identity.

10 Additional Concerns

10.1 The Extension to Higher Dimensions

The extension of the counterexample to dimensions $2n > 4$ relies on a result about symplectic 2-products from [6]:

> "For any convex domains $X \subset \mathbb{R}^{2m}$ and $Y \subset \mathbb{R}^{2n}$ satisfying $c_{\text{EHZ}}(X) = c_{\text{EHZ}}(Y)$, one has $\text{Sys}(X \otimes_2 Y) = \text{Sys}(X) \cdot \text{Sys}(Y)$."

This is applied with $X = K \times T$ and $Y = B^{2n-4}(\sqrt{c_{\text{EHZ}}(K \times T)/\pi})$. Several issues arise:

1. The result from [6] is stated as Theorem 1.2 therein. One must verify that the hypotheses of this theorem are satisfied, including any smoothness or strict convexity assumptions. The domain $K \times T$ is neither smooth nor strictly convex.

2. The 2-product $X \otimes_2 Y$ of a polytope and a ball is not itself a polytope (as the authors acknowledge), but it must still be convex for Viterbo's conjecture to apply. The convexity of p-products is known for $p \geq 1$, but the interaction with the capacity computation may require additional verification.

3. The capacity of the ball is $c_{\text{EHZ}}(B^{2n-4}(r)) = \pi r^2$, which gives $\mathrm{Sys}(B^{2n-4}(r)) = 1$. The multiplicativity then gives $\mathrm{Sys}(X \otimes_2 Y) = \mathrm{Sys}(X) > 1$. This step appears correct *if* the base case ($n = 2$) is established and *if* the cited theorem applies. But since the base case is precisely what is in question, the extension provides no independent support.

10.2 The Claim About 2-Bounce Trajectories

The authors claim that all 2-bounce trajectories from a vertex v_k to the opposite edge have the same T°-length as the diagonal. This relies on the orthogonality relation:

> "Since $v_{k+3}-v_{k+2}$ is orthogonal to $w_{k+1}-w_{k-1}$, the above expression is independent of λ."

Proposition 10.1. *The claim that all 2-bounce trajectories from v_k to the opposite edge have the same T°-length requires verification that the active support vertices are correctly identified for all $\lambda \in [0, 1]$.*

Proof. The authors write:

$$h_T(v_k - \widetilde{q}) + h_T(\widetilde{q} - v_k) = \langle v_k - \widetilde{q}, w_{k+1}\rangle + \langle \widetilde{q} - v_k, w_{k-1}\rangle,$$

where $\widetilde{q} = \lambda v_{k+2} + (1 - \lambda)v_{k+3}$.

For this to hold, one needs:

- $h_T(v_k - \widetilde{q}) = \langle v_k - \widetilde{q}, w_{k+1}\rangle$, i.e., $v_k - \widetilde{q} \in C_{k+1}$,
- $h_T(\widetilde{q} - v_k) = \langle \widetilde{q} - v_k, w_{k-1}\rangle$, i.e., $\widetilde{q} - v_k \in C_{k-1}$.

As λ varies from 0 to 1, the vector $v_k - \widetilde{q}$ rotates. The authors need to verify that this rotation stays within C_{k+1} for all λ. If it crosses into C_{k+2} or C_k for some λ, the claimed identity fails.

A similar verification is needed for $\widetilde{q} - v_k$ staying within C_{k-1}.

While these conditions may well be satisfied by the specific geometry of the regular pentagon, they constitute a geometric claim that requires explicit verification rather than being "immediate." □

11 Conclusion

We have identified five substantial logical gaps in the proof of the central Proposition of [1], upon which the claimed counterexample to Viterbo's volume-capacity conjecture entirely depends.

1. The combinatorial classification of non-translatable 3-vertex configurations is incomplete (Section 3).
2. The identification of active support vertices across deformation families is not exhaustive (Section 4).
3. The triangle inequality reduction does not preserve admissibility (Section 5).

4. The symmetry reduction is applied to an incomplete base case (Section 6).
5. The partial minimization does not establish global minimality (Section 7).

Each of these gaps, individually, could lead to an error in the claimed capacity value. Given that the margin by which the systolic ratio exceeds unity is only $\sim 4.72\%$ (corresponding to a $\sim 2.34\%$ margin in the capacity value), even a small correction would eliminate the counterexample.

We emphasize that we do not claim Viterbo's conjecture is true. Nor do we claim that the capacity value computed in [HKO] is necessarily wrong. What we do claim is that **the proof as presented is not logically complete**, and therefore the main theorem—that Viterbo's conjecture admits a counterexample—is not established by the arguments given.

A complete proof of Proposition [HKO] would require:

- A rigorous and exhaustive classification of all non-translatable configurations of 2 and 3 points on the boundary of the regular pentagon.
- A complete analysis of the piecewise-linear structure of the T°-length functional over all such configurations, including precise identification of active support vertices in each region.
- A valid comparison argument that respects the admissibility constraint throughout.
- Either a direct global minimization (e.g., via exhaustive piecewise-linear analysis) or a duality argument establishing a tight lower bound.

Until such a proof is provided, the status of Viterbo's volume-capacity conjecture remains, in our assessment, open.

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